$D(\lambda)$ lying above the contour Λ, and also over the roots of $n_3(-\lambda)$.

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DETERMINATION OF FREQUENCIES OF NATURAL VIBRATIONS OF CIRCULAR PLATES

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O. K. AKSENTIAN and T. N. SELEZNEVA
(Rostov-on-Don)
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A method to construct an asymptotic process to find the axisymmetric vibration frequencies of a circular plate is proposed. Cases of symmetric vibrations relative to the middle surface (tension-compression vibrations) and of antisymmetric (bending) vibrations are considered.

The asymptotic process for a plate with free endfaces has been studied in detail under mixed boundary conditions on the side surface. This problem can be considered as a model on which the practical convergence of the method proposed is analyzed and the accuracy of finding the frequencies at each step of the process is estimated. Furthermore, problems about the natural vibrations of a circular plate under other boundary conditions on the side surface, hinged-support and rigidly fixing, are solved by the proposed method.

The purpose of this investigation is to develop a method of determining the natural vibration frequencies of a "medium" thickness plate. The question of finding the higher frequencies, even for thin plates, as well as the lowest vibration frequencies of medium thickness plates cannot be solved within the framework of existing applied theories. Hence, it is interesting to formulate a sequence of approximate theories which would permit determination of any, previously assigned, number of the first frequencies with sufficient accuracy for medium thicknesses.

1. The problem concerns the natural vibrations of a circular plate under the following boundary conditions:

$$\sigma_z = \tau_{rz} = 0, \quad z = \pm h \tag{1.1}$$

$$u_r = \tau_{rz} = 0, \quad r = a \tag{1.2}$$

Here a is the plate radius and $2h$ is its thickness. Let us construct the solution in the form

$$u_r = U(\rho, \zeta)e^{i\omega t}, \quad w = W(\rho, \zeta)e^{i\omega t}, \quad \rho = \frac{r}{a}, \quad \zeta = \frac{z}{h} \tag{1.3}$$

Satisfying the system of Lamé differential equations and the boundary conditions (1.1) by using the symbolic method of Lur'e [1], we obtain the differential equation (1.6) for the resolving function ψ connected with the displacement vector components by the following relationships:

$$U = \frac{\partial}{\partial \rho}\left[2B \sin \lambda B \cos \lambda A\zeta - (\Delta^2 + A^2)\frac{\sin \lambda A}{A} \cos \lambda B\zeta\right]\psi \qquad (1.4)$$

$$W = \left[(\Delta^2 + A^2) B \sin \lambda B\zeta \frac{\sin \lambda A}{A} - 2\Delta^2 B \sin \lambda B \frac{\sin \lambda A\zeta}{A}\right]\psi$$

in the symmetric vibrations case and

$$U = \frac{\partial}{\partial \rho}\left[2A \cos \lambda B \sin \lambda A\zeta - (\Delta^2 + A^2)\frac{\sin \lambda B\zeta}{B} \cos \lambda A\right]\psi \qquad (1.5)$$

$$W = [2\Delta^2 \cos \lambda A\, \zeta \cos \lambda B - (\Delta^2 + A^2) \cos \lambda B\, \zeta \cos \lambda A]\psi$$

in the antisymmetric vibrations case. Here

$$\Delta^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}, \quad A^2 = \Delta^2 + \Omega^2, \quad B^2 = \Delta^2 + \frac{1-2\sigma}{2(1-\sigma)}\Omega^2$$

$$\Omega^2 = \frac{\omega^2 a^2 \rho_1}{G}, \quad \lambda = \frac{h}{a}$$

(G is the shear modulus, σ the Poisson's ratio, ρ_1 the density of the plate material)

$$\left\{\frac{\operatorname{tg} \lambda A}{\operatorname{tg} \lambda B} - \left[\frac{(\Delta^2 + A^2)^2}{4\Delta^2 AB}\right]^{\pm 1}\right\}\psi = 0 \qquad (1.6)$$

The upper sign in (1.6) corresponds to bending vibrations, and the lower sign to tension-compression vibrations.

Let us briefly describe the construction of the exact solution of the problem. We use the available arbitrariness in the homogeneous solution of (1.4), (1.6) and (1.5), (1.6) to satisfy the boundary conditions (1.2) by using the Betti theorem, say. We seek the solution of (1.6) in the class of functions satisfying the equation

$$\Delta^2\psi = \mu\psi \qquad (1.7)$$

$$\psi = M I_0(\sqrt{\mu}\rho) \qquad (1.8)$$

Substituting (1.7) into (1.6), we have the Lamb equation for μ

$$\frac{\operatorname{tg} \lambda\alpha}{\operatorname{tg} \lambda\beta} - \left[\frac{(\mu + \alpha^2)^2}{4\mu\alpha\beta}\right]^{\pm 1} = 0 \qquad (1.9)$$

$$\alpha^2 = \mu + \Omega^2, \quad \beta^2 = \mu + (1-2\sigma)\,\Omega^2 / 2(1-\sigma)$$

Summing solutions of the form (1.8) over roots μ_n of (1.9), we have

$$\psi = \sum_{n=1}^{\infty} M_n I_0(\sqrt{\mu_n}\rho) \qquad (1.10)$$

Satisfying the boundary conditions (1.2), we reduce the problem to an infinite system of homogeneous algebraic equations with a diagonal matrix in M_n

$$M_n F(\mu_n, \Omega, \sigma) I_1(\sqrt{\mu_n}) = 0 \quad (n = 1, 2, \ldots) \qquad (1.11)$$

Here $F(\mu_n, \Omega, \sigma)$ is the derivative of the right side of the Lamb equation with respect to η_n, where $\eta_n^2 = \mu_n$. Since the Lamb equation has no nonzero multiple roots [2] in the range of vibration Ω under consideration, then $F(\mu_n, \Omega, \sigma) \neq 0$. The system (1.11) yields

$$I_1(\sqrt{\mu_n}) = 0 \quad (n = 1, 2, \ldots) \qquad (1.12)$$

Substituting the roots of the Bessel function into (1.9), we obtain the frequency equations, later called "exact" equations.

We proceed to construct the asymptotic process to find approximately the natural vibration frequencies of a circular plate with the boundary conditions (1.2). Let us again start from the representation (1.10) for the function ψ. Taking account of (1.3) — (1.5), the boundary conditions (1.2) yield

$$\frac{\partial}{\partial\rho}(\Delta^2 + A^2)[\cos\lambda B\cos\lambda A\zeta - \cos\lambda A\cos\lambda B\zeta]\psi\big|_{\rho=1} = 0 \qquad (1.13)$$

$$\frac{\partial}{\partial\rho}\left[2A\cos\lambda B\sin\lambda A\zeta - (\Delta^2 + A^2)\cos\lambda A\,\frac{\sin\lambda B\zeta}{B}\right]\psi\Big|_{\rho=1} = 0$$

in the bending vibrations case, and

$$\frac{\partial}{\partial\rho}\left[2B\sin\lambda B\cos\lambda A\zeta - (\Delta^2 + A^2)\,\frac{\sin\lambda A}{A}\cos\lambda B\zeta\right]\psi\Big|_{\rho=1} = 0 \qquad (1.14)$$

$$\frac{\partial}{\partial\rho}\left[(\Delta^2 + A^2)\frac{\sin\lambda A}{A}\,B\sin\lambda B\zeta - B\sin\lambda B\,\frac{\sin\lambda A\zeta}{A}\right]\psi\Big|_{\rho=1} = 0$$

in the tension-compression vibrations case.

The asymptotic process proposed is the following. Asymptotic expansions are constructed for the roots of (1.9) in powers of λ which permit evaluation of any number of the first roots μ_n with a previously assigned degree of accuracy. At this stage of the asymptotic process the function ψ will then be

$$\psi = \sum_{n=1}^{2N} M_n I_0(\sqrt{\mu_n}\rho) \qquad (1.15)$$

in the antisymmetric vibrations case and

$$\psi = \sum_{n=1}^{2N-1} M_n I_0(\sqrt{\mu_n}\rho) \qquad (1.16)$$

in the symmetric vibrations case, where N is the number of the approximation (N = 1, 2, 3, ...).

The left sides of the boundary conditions (1.13) are decomposed in power series in ζ

$$A_{11}\psi\,|_{\rho=1} + A_{12}\psi\,|_{\rho=1}\zeta^2\lambda^2 + \ldots = 0 \qquad (1.17)$$

$$A_{21}\psi\,|_{\rho=1}\zeta\lambda + A_{22}\psi\,|_{\rho=1}\zeta^3\lambda^3 + \ldots = 0$$

where A_{ij} are differential operators of infinite order determined easily from (1.13). The boundary conditions (1.14) are also decomposed into series similarly. The appropriate differential operators are later marked with an asterisk in the symmetric vibrations case.

Equating coefficients of several of the first powers of ζ in (1.17) to zero, we obtain a finite set of boundary conditions for the function ψ. The number of boundary conditions to be satisfied at each stage of the process should agree with the number of first roots of μ_n taken for the Lamb equation (with the number N).

Thus, we have the following boundary conditions:

$$A_{ij}\psi|_{\rho=1} = 0 \quad (i = 1, 2;\ j = 1, 2, \ldots N) \tag{1.18}$$

at the N-th stage of the asymptotic process in the bending vibrations case and

$$\begin{aligned} A_{1j}{}^*\psi|_{\rho=1} &= 0 \quad (j = 1, 2, \ldots, N) \\ A_{2k}{}^*\psi|_{\rho=1} &= 0 \quad (k = 1, 2, \ldots, N-1) \end{aligned} \tag{1.19}$$

un the tension-compression vibrations case.

Expansions of the left sides of (1.13) in Fourier series or in series of Legendre polynomials can be constructed in place of (1.17). A numerical analysis carried out showed that the latter method is most efficient.

Now, let us determine the natural frequencies of the problems constructed successively (for $N = 1, 2, \ldots$). Substituting (1.15) into (1.18), (1.16) into (1.19) and equating the determinant of the homogeneous systems of linear algebraic equations in M_n to zero, we obtain an equation to determine the natural frequencies at the N-th stage.

We present the asymptotic expansions of the roots of the Lamb equation. In the case of antisymmetric vibrations, the Lamb equation has two roots on the order of $1/\lambda$ and a countable set of roots on the order of $1/\lambda^2$

$$\mu_n = \frac{\Omega}{\lambda}\left[(-1)^n \sum_{i=0,2,4,\ldots}^{\infty} \lambda^i\Omega^i C_i + \sum_{i=1,3,\ldots}^{\infty} \lambda^i\Omega^i C_i\right] \quad (n = 1, 2) \tag{1.20}$$

$$\mu_k = \frac{1}{\lambda^2}\sum_{i=0}^{\infty} F_{ki}\Omega^{2i}\lambda^{2i} \quad (k = 3, 4, \ldots), \quad \mu_{p+1} = \bar{\mu}_p \quad (p = 3, 5, \ldots) \tag{1.21}$$

Recursion systems are constructed to find the coefficients C_i and F_{ki}, where F_{k0} are the roots of the equation

$$\sin 2\sqrt{F_{k0}} - 2\sqrt{F_{k0}} = 0$$

In the tension-compression vibrations case, the Lamb equation has one root on the order of λ^0 and a countable set of roots on the order of $1/\lambda^2$

$$\mu_1{}^* = \Omega^2\sum_{i=0}^{\infty} \lambda^{2i}\Omega^{2i}C_i{}^*$$

$$\mu_k{}^* = \frac{1}{\lambda^2}\sum_{i=0}^{\infty} \lambda^{2i}\Omega^{2i}F_{ki}{}^* \quad (k = 2, 3, \ldots), \quad \mu_{p+1}^* = \bar{\mu}_p{}^* \quad (p = 2, 4, \ldots)$$

Recursion systems are obtained for the coefficients $C_i{}^*$ and $F_{ki}{}^*$, where $F_{k0}{}^*$ are the roots of the equation

$$\sin 2\sqrt{F_{k0}^*} + 2\sqrt{F_{k0}^*} = 0$$

In the bending vibrations case μ_k $(k = 3, 4, \ldots)$ are complex in the range of variation Ω under consideration, therefore, $I_1(\sqrt{\mu_k}) \neq 0$. Finally, the frequency equation is

$$I_1\left(\left\{\frac{\Omega}{\lambda}\left[(-1)^n \sum_{i=0,2,4,\ldots}^{\infty} \lambda^i\Omega^i C_i + \sum_{i=1,3,\ldots}^{\infty} \lambda^i\Omega^i C_i\right]\right\}^{1/2}\right) = 0 \quad (n = 1, 2)$$

Let us compare this with the results of the Mindlin [3] and classical [4] theories. The relationship between μ and Ω can be represented by an asymptotic expansion of the form

$$\mu_n = \frac{\Omega}{\lambda}\left[(-1)^n C_0 + \lambda\Omega C_1(1 + \varepsilon_1) + (-1)^n \lambda^2\Omega^2 C_2(1 + \varepsilon_2) + \ldots\right]$$
$$(n = 1, 2)$$

in the Mindlin theory of bending vibrations. The constants ε_1, ε_2 depend on the body material and for $\sigma = 1/3$ we have $\varepsilon_1 = 0.056$ and $\varepsilon_2 = 0.36$. Therefore, the first term of the asymptotic expansion for μ from the Mindlin theory agrees (as in the classical theory) with the corresponding term of the expansion (1.20); the coefficient of λ differs by 5.6% and of λ^2 by 36% from the corresponding coefficients of the expansion (1.20). Therefore, the refinement given by the Mindlin theory is that this theory takes account of the second term of the asymptotic series (1.20) in the frequency equation (this term is determined sufficiently exactly according to Mindlin). Hence, it can be expected that for small λ this theory will permit a better determination of the frequency than will the classical theory. For sufficiently large λ when the next terms of the expansion must be retained in the series, the Mindlin theory will not yield good results.

The frequency equation in the tension-compression case is

$$I_1\left(\Omega\left[\sum_{i=0}^{\infty} \lambda^{2i}\Omega^{2i}C_i^*\right]^{1/2}\right) = 0$$

The relationship between μ and Ω can be represented by the following asymptotic expansion

$$\mu_1^* = \Omega^2\left[C_0^* + \lambda^2\Omega^2 C_1^*\,(1 - e_1) + \lambda^4\Omega^4 C_2^*\,(1 - e_2) + \ldots\right]$$

in the Mindlin theory of symmetric vibrations [5], where e_1, e_2 depend on the body material and we have $e_1 = 0.189$ and $e_2 = 0.574$, for $\sigma = 1/3$. Therefore, the first term of the asymptotic expansion for μ_1^* agrees with the true value, the second differs by 18.9% and the third by 57.4%. The lesser accuracy of the Mindlin theory in determining the tension-compression vibration frequencies than the bending vibration frequencies is explained by the greater error in determining the second coefficient of the expansion as compared with the bending vibrations case.

A computation of the roots of the frequency equations at different stages of the approximation and of the exact equation for $\sigma = 1/3$ and for different values of the parameter λ was conducted on the electronic digital computer "Odra-1204".

Calculations showed that for small λ ($\lambda \sim 0.1$) the asymptotic method permits determination of more than ten first frequencies with an error not exceeding 0.5% in the bending vibrations case. Table 1 illustrates the practical convergence of the proposed asymptotic method for $\lambda = 0.1$, 0.5. Values of the frequencies obtained by the asymptotic method are presented in row I, the exact values of the frequencies in II, the values of the frequencies calculated by the Mindlin and classical theories, respectively, in rows III and IV.

A numerical analysis of the convergence of the asymptotic method was carried out for different p (p is the number of terms in the expansion (1.20) and n is the number of the frequency in increasing order). For Table 1 we use $p = 8$. For larger values of λ ($\lambda \geqslant 0.2$) the asymptotic method permits determination of six - seven first frequencies with an error not exceeding 1 — 2%. For small λ ($\lambda < 0.1$), the asymptotic method permits determination of more than ten first frequencies with error not exceeding 2% in the symmetric vibrations case. For $\lambda \geqslant 0.1$ the convergence of the method is rather worse than in the model problem of antisymmetric vibrations. The results of computing the natural frequencies for $\lambda = 0.4$ and $p = 6$ are presented in Table 2. Furthermore, the bending vibration frequencies of a circular plate were calculated by the method proposed under different boundary conditions on the side surface.

2. Let us consider the axisymmetric problem of the bending vibrations of a circular plate with hinged support on the side surface

$$\sigma_z = \tau_{rz} = 0, \quad z = \pm h$$
$$\sigma_r = w = 0, \quad r = a$$

The boundary conditions on the side surface can be represented as

$$\left\{\frac{\partial^2}{\partial\rho^2}\left[2A\sin\lambda A\zeta\cos\lambda B - (\Delta^2 + A^2)\frac{\sin\lambda B\zeta}{B}\cos\lambda A\right] + (\Delta^2 + A^2)\frac{\sigma\Omega^2}{2(1-\sigma)}\frac{\sin\lambda B\zeta}{B}\cos\lambda A\right\}\psi\Big|_{\rho=1} = 0$$

$$[2\Delta^2\cos\lambda A\zeta\cos\lambda B - (\Delta^2 + A^2)\cos\lambda B\zeta\cos\lambda A]\,\psi\,|_{\rho=1} = 0$$

The left sides of these repationships are expanded in power series of ζ. In the first stage of the asymptotic process we have

$$\psi = M_1 I_0(\sqrt{\mu_1}\,\rho) + M_2 I_0(\sqrt{\mu_2}\,\rho)$$

$$\left\{\frac{\partial^2}{\partial\rho^2}[2A^2\cos\lambda B - (\Delta^2 + A^2)\cos\lambda A] + (\Delta^2 + A^2)\frac{\sigma\Omega^2}{2(1-\sigma)}\cos\lambda A\right\}\psi\Big|_{\rho=1} = 0$$

$$[2\Delta^2\cos\lambda B - (\Delta^2 + A^2)\cos\lambda A]\,\psi\,|_{\rho=1} = 0$$

We hence obtain a homogeneous system of linear algebraic equations in M_n

$$\sum_{i=1}^{2}\left\{[2\alpha_i^2\cos\lambda\beta_i - (\mu_i + \alpha_i^2)\cos\lambda\alpha_i][\mu_i I_0(\sqrt{\mu_i}) - \sqrt{\mu_i}I_1(\sqrt{\mu_i})] + (\mu_i + \alpha_i^2)\frac{\sigma\Omega^2}{2(1-\sigma)}\cos\lambda\alpha_i I_0(\sqrt{\mu_i})\right\}M_i = 0 \qquad (2.1)$$

$$\sum_{i=1}^{2}[2\mu_i\cos\lambda\beta_i - (\mu_i + \alpha_i^2)\cos\lambda\alpha_i]\,I_0(\sqrt{\mu_i})\,M_i = 0$$

$$\alpha_i^2 = \mu_i + \Omega^2, \quad \beta_i^2 = \mu_i + (1-2\sigma)\,\Omega\,/\,2\,(1-\sigma)$$

Substituting the values of μ_1, μ_2 from (1.20) into the determinant of this system and then equating the determinant to zero, we obtain the frequency equation in the first stage In the second stage we have

$$\psi = \sum_{i=1}^{4} M_i I_0(\sqrt{\mu_i}\rho) \qquad (2.2)$$

$$\sum_{i=1}^{4}\left\{[2\alpha_i^2\cos\lambda\beta_i - (\mu_i + \alpha_i^2)\cos\lambda\alpha_i][\mu_i I_0(\sqrt{\mu_i}) - \sqrt{\mu_i}I_1(\sqrt{\mu_i})] + (\mu_i + \alpha_i^2)\frac{\sigma\Omega^2}{2(1-\sigma)}\cos\lambda\alpha_i I_0(\sqrt{\mu_i})\right\}M_i = 0$$

$$\sum_{i=1}^{4}[2\mu_i\cos\lambda\beta_i - (\mu_i + \alpha_i^2)\cos\lambda\alpha_i]\,I_0(\sqrt{\mu_i})\,M_i = 0$$

$$\sum_{i=1}^{4}\Big\{2\alpha_i^4\cos\lambda\beta_i-(\mu_i+\alpha_i^2)\beta_i^2\cos\lambda\alpha_i]\,[\mu_i I_0(\sqrt{\mu_i})-\sqrt{\mu_i}\,I_1(\sqrt{\mu_i})]+$$
$$(\mu_i+\alpha_i^2)\frac{\sigma\Omega^2}{2(1-\sigma)}\beta_i^2\cos\lambda\alpha_i I_0(\sqrt{\mu_i})\Big\}M_i=0$$
$$\sum_{i=1}^{4}[2\mu_i\alpha_i^2\cos\lambda\beta_i-(\mu_i+\alpha_i^2)\beta_i^2\cos\lambda\alpha_i]\,I_0(\sqrt{\mu_i})\,M_i=0$$

Similarly to the above, we obtain an equation to determine the natural frequencies in the second stage from (2.2) and (1.20).

Table 1

λ	0.1						0.5				
n	1	2	3	...	9	10	1	2	...	5	6
I	1.336	3.815	6.770		20.08	22.58	3.158	6.330		9.472	10.02
II	1.336	3.815	6.770		20.11	22.59	3.158	6.324		9.397	10.12
III	1.329	3.769	6.647		20.51	21.81	3.066	6.096		11.93	12.68
IV	1.468	4.921	10.35				7.341	24.61		135.6	192.4

Table 2

n	I	II	III	IV
1	5.562	5.485	6.016	6.637
2	7.447	7.448	9.165	12.15
3	8.391	8.157	9.945	17.62
4	10.04	9.868	11.90	23.08

Table 3

	I		III
n \ N	1	2	
1	1.269	1.269	1.220
2	4.379	4.372	4.245
3	6.468	6.691	6.603
4	7.545	7.536	7.280
5	10.16	10.26	10.25
6	10.71	10.82	11.04

Table 4

	I		III	IV
n \ N	1	2		
1	1.692	1.732	1.763	3.065
2	4.430	4.506	4.406	11.93
3	7.546	7.563	7.270	26.73
4	8.385	8.313	8.750	47.75
5	10.64	10.66	10.28	
6	12.37	12.32	13.62	

Table 5

	I			IV
n \ N	1	2	3	
1	2.463	2.460	2.460	4.538
2	4.754	4.750	4.750	19.26
3	6.372	6.162	6.215	43.91
4	8.629	8.057	8.038	78.45
5	9.895	9.174	9.071	122.8

Let us note that the principal second order minor in the determinant of the system (2.2) agrees with the determinant of the system (2.1). The results of computing the natural frequencies for $\lambda = 0.3$ are given in Table 3. Values of the frequencies for $N = 1, 2$ (N is the number of the approximation) are presented in column I.

3. Moreover, the problem of the bending vibrations of a circular plate whose side surface is rigidly clamped

$$\sigma_z = \tau_{rz} = 0, \quad z = \pm h \tag{3.1}$$

$$u_r = w = 0, \quad r = a \tag{3.2}$$

is solved by the asymptotic method. The condition (3.2) can be represented as

$$\frac{\partial}{\partial\rho}\left[2A \sin\lambda A\zeta \cos\lambda B - (\Delta^2 + A^2)\frac{\sin\lambda B\zeta}{B}\cos\lambda A\right]\psi\Big|_{\rho=1} = 0$$

$$[2\Delta^2 \cos\lambda A\zeta \cos\lambda B - (\Delta^2 + A^2)\cos\lambda B\zeta \cos\lambda A]\,\psi\,|_{\rho=1} = 0$$

Similarly to the above, the frequency equations in the first two stages of the approximation were obtained. The natural frequencies are presented in Table 4 for $\lambda = 0.3$. The column notation is the same as before.

The problem of antisymmetric and symmetric vibrations of a circular plate with a free side surface has been examined. The boundary conditions were satisfied by expanding them in series of Legendre polynomials. The natural frequencies of the antisymmetric vibrations are presented in Table 5 for $\lambda = 0.5$.

The deductions about the convergence of the asymptotic method for the last three problems are the same as in the model problem of antisymmetric vibrations.

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